

# Quantum Simplicial Dynamics

David Finkelstein  
Georgia Institute of Technology

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## Abstract

Present-day quantum field theory can be regularized by a decomposition into quantum simplices. This replaces the infinite-dimensional Hilbert space by a high-dimensional spinor space and singular canonical Lie groups by regular spin groups. It radically changes the uncertainty principle for small distances. Gaugeons, including the gravitational, are represented as bound fermion-pairs, and space-time curvature as a singular organized limit of quantum non-commutativity.

Keywords: Quantum logic, quantum set theory, quantum gravity, quantum topology, simplicial quantization.

## 1 Simplicial quantization

### 1.1 Quantization and de-radicalization

One way to regularize the Standard Model and general relativity while preserving their agreement with empirical particle spectra and selection rules is to analysis them into finite quantum simplicial elements. This might be a physical regularization, not a formal one.

A quantum theory represents its unmixed system input and outtake processes by rays in a space of what Heisenberg called “probability vectors”. A “probability-amplitude vector” is more accurate; a “ $\psi$  vector” or simply “a  $\psi$ ” is quicker. If the  $\psi$  space is finite-dimensional, the operators of the theory have only finite spectra, and divergences do not occur. Such a quantum theory is called *regular* [5].

Call a Lie algebra or group regular or singular according to whether its Cartan-Killing form is so. Semisimple Lie algebras are regular; canonical Lie algebras are singular.

Quantum theories today use infinite-dimensional  $\psi$ 's for unitary representations of singular or non-compact Lie algebras.

The singular Lie algebras include the canonical/Bose-Einstein Lie algebra

$$\mathfrak{h}(M) = \text{alg}(x^\mu, y_\mu, i) : [x^\mu, y_{\mu'}] = \delta_\mu^{\mu'}, \quad i \leq \mu, \mu' \leq M, \quad (1)$$

of classical space-time coordinates and differentiators, ordinary or gauge-covariant; of quantum coordinates and their canonical momenta in general; of the boson fields in

particular; and the functional algebras of gauge theory, including the diff of general covariance.

The non-compact Lie groups include the Lorentz group. Its Hilbert space representations are singular but its spinor representations are regular.

The radicals of the singular algebras indicate how singular they are. Canonical quantization reduced the radical dimensions of some Lie algebras of classical physics from  $\infty$  to 1 by decontraction. Here the remaining singular Lie algebras are further reformed into simple subalgebras of  $\text{spin}(N, N)$  for some large finite  $N$  determined by the number of quantum events in the system history. This eliminates the radicals.

The limit  $N \rightarrow \infty$  is not a contraction in the usual sense. It goes by discrete steps instead of a homotopy. Yet in a weak sense still to be made precise, the step from  $N$  to  $N + 1$  is small when  $N$  is large. If  $N$  is large enough, for example, the difference between systems of  $N$  and  $N + 1$  quanta will be experimentally unresolvable by present means. Let us term a succession of small steps toward regularity a *reformation* whether they are discrete steps or infinitesimal.

Quantizations have required “painful renunciations”, as Bohr put it. Now the ideas of space-time, Hilbert space, and a fundamental theory are renounced. A typed and graded spinor space  $\mathbb{S}$  replaces and elaborates the usual Hilbert space. Its indefinite metric has a familiar physical interpretation, essentially given by Dirac (§1.2).

Classical systems with ancestrally finite sets for their sample spaces (= phase spaces) are intrinsically finite in their properties (§1.2). Finite sets are also simplices, whose vertices are their elements and whose points are the convex statistical mixtures of their vertices. Quantum simplices replace the classical statistical mixture by quantum superposition (§1.2).

Vertices of a simplex may themselves be simplices of a deeper level. Call a simplex a *complex* when we wish to emphasize that it is not only finite but ancestrally finite, finite all the way down.

The quantum complex of §1.2 is regular, with its  $\psi$ 's in a spinor space, finite dimensional by definition, and seems rich enough to represent any finite quantum structure and any of the experimental transformation groups of quantum physics as closely as necessary. It seems closer to experiment to use  $\psi$ 's in a spinor than in a Hilbert space.

From a logic of yes-or-no questions about one individual one can construct a logic of how-many questions about a quantity of such individuals. Such a construction is called *quantification* after William Hamilton, ca. 1850. The higher-order algebra of sets (or classes) is a quantification theory. It transforms a theory with sample space  $S$  into a quantified theory with sample space  $2^S$ , the power set of  $S$ .

As von Neumann pointed out, when Heisenberg replaced the commutative algebra of dynamical variables by his non-commutative algebra of dynamical variables, he effectively revised the first-order logic of physical systems; for predicates are merely two-valued dynamical variables. Quantum theory replaced Boolean logic by a projective logic.

This destroyed classical quantification theory at its Boolean base and necessitated a quantum replacement. Several quantum quantifications were swiftly constructed; but from scratch, as if classical quantification had never existed. Classical quantifications include such assemblies as combinations and permutations, sets and sequences. Quan-

tum sets were instead called Fermi-Dirac ensembles. Quantum sequences were called Maxwell-Boltzmann ensembles. These quantifications convert each one-quantum  $\psi$  into an annihilation/creation (or A/C) operator  $\hat{\psi}$ , acting on many-quantum  $\psi$  vectors, and generating an algebra that determines the statistics.

Call a classical sequence  $(e, e, \dots, e)$  of any number of equal elements, a *sib*. This construct was too useless to merit a name before the advent of the quantum theory. A quantum sib, however, is a Bose-Einstein ensemble, whose every  $\psi$  is a superposition of tensor powers like  $\vec{e} \otimes \dots \otimes \vec{e}$ , where  $\vec{e}$  indicates a  $\psi$  vector.

All the classical modes of quantification mentioned are easily expressed in terms of power sets, but the converse is not true. Moreover, the only regular quantification theory among them is the quantum set, the Fermi-Dirac ensemble, whose  $\psi$  space is a Grassmann algebra. This suggests that if we choose to start with just one kind of quantification, even as a toy theory, we ought to try the Fermi-Dirac kind first. Let us do so.

## 1.2 The spinor space $\mathbb{S}$

Since space-time belongs to a deeper logical type than the fields that are functions on it, their joint quantization calls for a quantum theory of several levels.

*Unitization* (Glaserfeld's term) is the bracing operation

$$\iota : a \mapsto \{a\} \equiv \iota a \equiv \bar{a}. \quad (2)$$

Unlimited iteration of the classical operations  $1$ ,  $\iota$  and  $\vee$  generates an infinite set  $\widehat{\mathbb{S}}$  of classical (finite) complexes.

Correspondingly, the  $\psi$  space  $\mathbb{S}$  of the generic quantum complex is finitely generated by  $+$  and  $\mathbb{R}$  for quantum superposition, the unitization operator or unitizer  $\iota$ , subject to linearity, and the Grassmann product  $\vee$ .

Iterated unitization is standard in classical thought; Peano used it to generate the natural numbers. It already occurs to a limited extent at several junctures in quantum physics today; as when we treat helium nuclei as point particles. There  $\iota$  is a phenomenological description of a more complex process of dynamical binding of four fermions. Likely  $\iota$  is phenomenological here too.

Write the set of all finite subsets of a set  $s$  as  $S = \bigvee s$  or  $2^s$ . Let us call the set  $s$  the *logarithm* of  $2^s$ . For example,  $\widehat{\mathbb{S}}$  is its own logarithm:  $\widehat{\mathbb{S}} = 2^{\widehat{\mathbb{S}}}$ .

. As usual, use the same symbol for corresponding classical and quantum constructs when context makes it clear which is meant. When necessary, designate a quantum correspondent of  $x$  by  $\vec{x}$ . For example, write the Grassmann or exterior algebra of a vector space  $\mathcal{V}$  as  $\bigvee \mathcal{V} \equiv \mathbf{2}^{\mathcal{V}}$ . This is the vector space spanned by Grassmann products of finitely many vectors in  $\mathcal{V}$ .

Like  $\widehat{\mathbb{S}}$ ,  $\mathbb{S}$  is its own logarithm,  $\mathbb{S} = \bigvee \mathbb{S} = 2^{\mathbb{S}}$ , and is doubly graded, by Grassmann grade  $g$  and Quine type  $T$ :

$$\mathbb{S} = \bigoplus_{g,T} \mathbb{S}^{(g)}[T] \quad (3)$$

The spinors of  $\mathbb{S}$  are used as  $\psi$ 's for quantum complexes.

These spinors carry many physical variables besides spin. For clarity, call a quantum complex a *plexon*, and call a  $\psi$  vector for a plexon a *plexor*. Spinors are special cases.

Since simplicial quantum theory replaces the Hilbert space of canonical quantum theory by a Grassmann algebra  $\mathbb{S}[E]$ , it is a graded (or “super-”) quantum theory of an extreme kind. In the terms of Bryce DeWitt, plexons are all soul and no body.

In nature, all elementary fermions seem to have spin 1/2. The systems represented in the Grassmann algebra  $\mathbb{S}$  are fermions. Therefore give each type  $\mathbb{S}$  the structure of a spinor space, not a Hilbert space (§2.3). The Cartan theory of spinors enables us to assign a spinor structure to every Grassmann algebra  $\mathbb{S}[T]$ . The underlying quadratic space is the direct sum

$$\mathcal{W}[T] := \text{Bi}\mathbb{S}^{(1)}[T] := \mathbb{S}^{(1)}[T] \oplus \text{Dual}\mathbb{S}^{(1)}[T]. \quad (4)$$

The quadratic space underlying the spinor space  $\mathbb{S}$  is the bipolar space of (14).

In homological algebra, a *complex*—a collection of simplices—is usually represented as a sum of simplices. But in  $\mathbb{S}$ , addition is mere quantum superposition, not collection. It does not increase the number of simplices but merely changes the possibility for one simplex. One assembles plexons into a plexon of higher type according to quantum logic; not by adding them but by multiplying their unitizations. The vertices of the resulting complex are the unitizations of the simplices of the type below. A complex is a simplex of simplices, not a sum of simplices.

### 1.3 Index conventions

$\mathbb{S}[T]$  := the subspace of  $\mathbb{S}$  consisting of all polyadics of type  $T$ .

$\text{Bi}\mathbb{S}[T] := \mathbb{S}^{(1)}[T] \oplus \text{Dual}\mathbb{S}^{(1)}[T]$ .

$1_t$  := a typical polyadic basis spinor in the Grassmann algebra  $\mathbb{S}[T]$ , indexed by a lower-case version of  $T$ , the symbol for the type.

$1_{\bar{t}}$  := a typical monadic basis vector in the first-grade subspace  $\mathbb{S}^{(1)}[T+1]$ :

$$1_{\bar{t}} := \overline{1_t} := \iota(1_t) =: 1_{(2^t)}. \quad (5)$$

$1_{\bar{t}}$  := a typical basis vector for the bispaces  $\text{Bi}\mathbb{S}[T+1]$ .




















































In the classical basis  $t$  is the serial number of  $1_t$ .  $t$  and  $\bar{t}$  take on hexp  $T$  values.  $\bar{t}$  takes on 2 hexp  $T$  values.

### 1.4 Bar codes

An algebraic *bar code* is easier to read and write than a graph of a multidimensional complex. Construct a bar code from a formula for the complex in the symbols 1,  $\iota$ , and  $\vee$ , by omitting the symbols 1 and  $\vee$  as usual, writing  $\iota x$  as the bar symbol  $\bar{x}$ , ordering factors by their serial numbers, and write superpositions as usual.  $\mathbb{S}$  is the vector space of quantum bar codes (Table 1).

Each bar code describes an oriented simplicial complex, whose vertices are its factors, oriented by the order of their multiplication. They must be read from the bottom up. Which bars stand for the same vertex is determined by what lies below them and

Table 1: Bar codes by rank  $r$  and serial number  $n$

6													...
	hexp 6	...											
5													...
	hexp 5	...											
4													...
	16	17	18	19	20	21	22	23	24	25	26	27	...
3													
	4	5	6	7	8	9	10	11	12	13	14	15	
2													
	2	3											
1													
	1												
0													
	0												
$r$	$\mathbf{1}_n$ $n$												

not by what lies above them. The bar codes constructed without quantum superposition, using only  $\vee$  and  $\iota$ , form a classical basis of  $\mathbb{S}$ . Their signs are fixed by serially ordering their monadic factors.

A  $g$ -adic is a simplex of exactly  $g$  vertices. Each vertex of a simplex is a segment of its bar code covered by a single bar, as long as necessary, representing a monadic.

The rank of a polyadic is the height in bars of its highest monadic factor.

A simplicial quantum theory of type  $T$  is one whose plexors have type  $\mathbb{S}[T] \subset \mathbb{S}$ .

Assume that there exist:

1. a cellular plexon type  $\mathbb{S}[C]$  that supports the Lie algebra of spin and the unitary charges.
2. an event type  $\mathbb{S}[E]$  that supports the orbital variables as well.
3. a field type  $\mathbb{S}[F] = \mathbb{S}[E + 1]$  that supports field variables as well.

Let us roughly estimate the event type  $\mathbb{S}[E]$  whose dimension  $N$  is large enough to pass for the Hilbert space of a fermion. Provisionally,  $C = 3$ ,  $\text{Dim } \mathbb{S}[C] = \text{hexp } 2 = 16$  might

suffice for the spin and charge of one cell. Then  $E = 4$  would imply  $N \leq \text{hexp } 4 = (2^{16})$ , far too small for a quasi-continuum of events; while  $E = 5$  implies  $N \leq \text{hexp } 5 = 2^{64K}$  (where  $K := 2^{10}$ ), far more than required.

The Poincaré group can be approximated within present experimental error, though non-uniformly, by a subgroup of  $\text{SO}(\mathbb{S}[5])$  but not  $\text{SO}(\mathbb{S}[4])$ . The Standard Model groups seem to be faithfully represented in  $\mathbb{S}[5]$  up to experimental accuracy.

Therefore assume tentatively that  $C = 3$ ,  $E = 5$ , and  $F = 6$ .

## 2 Space-time quantizations

### 2.1 Yang quantum space

Most physical theories today assume an absolute space-time, with many reference frames related by a relativity group. To simplify the Poincare-Heisenberg Lie algebra

$$\text{ph}(4) = \text{alg}(x^m, p_m, L_{m'm}, i) \quad (6)$$

of special relativistic mechanics, Yang dropped this assumption and adopted the decontraction  $\text{ph}(4) \leftarrow \text{so}(5, 1)$  [34]. This relativizes the split into position space and momentum space, and quantizes the imaginary  $i$  of the commutation relations. He retained a complex Hilbert space representation of infinity dimensions, however, with its separate central  $i$ .

Many reformations decontract the canonical Lie algebra  $\mathfrak{h}(1)$  to the sphere Lie algebra  $\text{so}(3)$ :

$$\text{so}(3) : \quad [p, q] = r, \quad [q, r] = \lambda p, \quad [r, p] = \lambda q, \quad \lambda \rightarrow 0, \quad (7)$$

replacing the canonical  $i$  by a variable  $r$ , and the singular Killing form by a regular one with negative-definite Killing form. Slightly different reformations de-contract  $\mathfrak{h}(1)$  to the hyperboloid Lie algebra:

$$\text{so}(2, 1) : \quad [p, q] = r, \quad [q, r] = \lambda p, \quad [r, p] = -\lambda q, \quad \lambda \rightarrow 0, \quad (8)$$

with indefinite Killing form. Another decontraction of the canonical Lie algebra  $\mathfrak{h}(1) := \text{alg}(q, p, i)$  of (1) acts on a spinor space instead of a Hilbert space, representing  $\text{so}(3)$  by dyadic spin operators in a Clifford algebra  $\mathbb{S}[2] \otimes \text{Dual } \mathbb{S}[2]$ , of type 2:

$$p = \gamma_{23}, \quad q = \lambda \gamma_{31}, \quad r = \lambda \gamma_{12}, \quad \lambda \rightarrow 0. \quad (9)$$

A unification of position and momentum occurs in other gauge theories in quantum space-time [16].

### 2.2 Feynman quantum space

Feynman proposed a spinorial space-time quantization that in retrospect fits into the neutral spinor space  $\mathbb{S}$ , not a Hilbert space  $\mathcal{H}$ . He wrote the coordinates as sums of many commuting Dirac matrix-vectors:

$$x^m \sim \gamma^m(1) \oplus \dots \oplus \gamma^m(N). \quad (10)$$

This was undoubtedly suggested by the relativistic proper-time equation of motion

$$\frac{dx^m}{d\tau} = \gamma^m \quad (11)$$

for a Dirac particle. These Dirac spin operators  $\gamma^m(n)$  belong to a quantum element of space-time. (10) concerns an assembly of  $N$  such elements with a Palev statistics, which had not been discovered yet. An implied quantum unit of time  $X$  is omitted from (10).

Feynman's proposal must be extended to represent momentum and  $i$ . This can be done within the same Dirac algebra, as noted by Marks [17]. For the Standard Model fermions,  $i\gamma^{4321}$  is  $\pm 1$ . This suggests that at a deeper level  $\gamma^{4321}$  performs some of the functions of  $i$ . If we set

$$\begin{aligned} p_m &\sim \gamma_{4321}(1)\gamma_m(1) \oplus \dots \oplus \gamma_{4321}(N)\gamma_m(N), \\ i &\sim \frac{\gamma_{4321}(1) \oplus \dots \oplus \gamma_{4321}(N)}{N}, \end{aligned} \quad (12)$$

then the commutator of  $\delta x$  and  $\delta p$  has a  $\gamma^{4321}$  where the canonical theory has  $i$ . (12) omits a quantum unit of energy

$$P = \frac{\hbar}{NX}, \quad (13)$$

required to balance the units. (12) defines a simplicial quantum space. In the limit of classical space-time,  $X \rightarrow 0$ ,  $P \rightarrow 0$ , and  $N \rightarrow \infty$ . The spin now serves as an element of space-time and momentum-energy, as well as angular momentum  $\delta L_{m'm} = \gamma_{m'm}/2$ . Soon its simplex will be enlarged to carry the unitary charges as well, and to describe entire events. When it is necessary in order to reduce confusion, call a spin with this enlarged physical interpretation and algebra, a quantum simplex, or *plexon*.

The space-time coordinates, Lorentz angular momenta, momenta, and  $i$  of a plexon of the cellular type correspond 1-1 to  $4 + 6 + 4 + 1 = 15$  traceless  $4 \times 4$  Dirac matrices in the real Dirac spinor space  $4\mathbb{R}$  of the plexon, soon to be enlarged. The commutator Lie algebra of  $\text{Cliff}(3\mathbb{R} \oplus 1\mathbb{R})$  is  $\mathfrak{sl}(4) \cong \mathfrak{so}(3,3)$  instead of the Yang  $\mathfrak{so}(5,1)$ . The Feynman quantum space has an automorphism  $\gamma^m \mapsto \gamma^{4321}\gamma^m$  of the Clifford algebra of polynomials in the  $\gamma^m$ ; it exchanges space-time with momentum-energy.

The Feynman quantum space is not represented in Hilbert space here but in a spinor space  $\mathbb{S}[E]$ , and so it is regular. If the extra two coordinates of  $\mathfrak{so}(3,3)$  are frozen out in the condensation that produces  $i$ , as assumed, their peculiar timelike signature seems to lead to no contradictions with macroscopic experience.

Each type  $\mathbb{S}[T]$  is a spinor space for a quadratic vector space  $\mathcal{W}[T]$ . The vector-wise unitization  $\iota^*\mathbb{S}[T] = \mathbb{S}^{(1)}[T+1]$  is the null semivector space for the next spinor space, of type  $T+1$ , depending on context. This context-dependence can be eliminated merely by explicitly tagging spinors and semivectors differently, so it seems harmless.

As spinor space, the spin group  $\text{Spin}[T]$  doubly covers the orthogonal group  $\text{SO}[T]$  of the *bipolar space*

$$\mathcal{W}[T] := \text{Dual } \mathbb{S}^{(1)}[T] \oplus \mathbb{S}^{(1)}[T] =: \text{Bi } \mathbb{S}^{(1)}[T] \quad (14)$$

with neutral *bipolar norm*  $b : \mathcal{W} \rightarrow \text{Dual } \mathcal{W}$  given by

$$\forall Q \in \mathbb{S}, \forall Q' \in \text{Dual } \mathbb{S}, W := Q + Q' :$$

$$bW \circ W := \|W\| = \|Q + Q'\| := Q' \circ Q. \quad (15)$$

The bipolar space is also called the quantum space [21]. A basis  $I_s$  for monadics in  $\mathbb{S}$  defines a basis  $I_{\bar{s}}$  for the bipolar vectors of  $\mathcal{W}$ . These in turn define a basis  $\gamma_{\bar{s}}$  for the first grade of the Clifford algebra over  $\mathcal{W}$ . Grassmann left-multiplication defines the first grade generator  $\gamma_s$  of the type- $T$  Clifford algebra  $\text{Cliff } \mathcal{W}^{(1)}[T]$ :

$$\forall \psi \in \mathbb{S}^{\lfloor T \rfloor}, I_s \in \mathcal{W}[T-1] : \gamma_s \psi := I_s \vee \psi. \quad (16)$$

And the dual basis elements  $\gamma^s$  are dually defined by Grassmann left-differentiation with respect to  $I_s$ .

The Dirac spin operators  $\gamma^m$  come from a low type  $C$  of this Clifford algebra. The Fermi-Dirac annihilation/creation operators  $\psi(x)$  belong to a high type  $E$  of the same Clifford algebra.

The indefinite metric  $b$  of  $\mathcal{W}$  gives rise to that of special relativity, which distinguishes the forbidden spacelike directions from the allowed timelike ones.

### 2.3 Neutral spinor form

Any spinor space has a bilinear *spinor form*, that is invariant under its spin group  $\text{Spin}[T]$ . Cartan called it  $C$ ; in electron theory it is often designated by  $\beta$ . For an irreducible spinor space,  $\beta$  is unique up to a numerical factor.

The addition of a quantum  $A$  to a system is equivalent to the removal of a quantum  $B = \text{Dual } A$  from the system. The anti-particle  $\bar{A}$  of a particle  $A$  with energy  $E$  is a dual particle with energy  $-E$ .

Consider a basis  $\gamma^{m\pm}$  of Clifford algebra generators that are anti-commuting square roots of  $\pm 1$ . Let  $\Pi_{\pm}$  be the product of the operators  $\gamma^{m\pm}$  that have square (say)  $\pm 1$ . Let  $1_s$  be a spinor basis in which the  $\gamma_{m+}$  are symmetric and the  $\gamma_{m-}$  are skew-symmetric. Let  $E = \sum 1_s \otimes 1_s$  be the Euclidean quadratic form that is diagonal in the basis  $1_s$ . Then two spinor forms, possibly proportional to each other, are

$$\beta_{\pm} = \Pi_{\pm} E \quad (17)$$

Write  $\beta[T]$  for the spinor form of  $\mathbb{S}[T]$ . Then  $\beta[2]$  is skew-symmetric and  $\beta[T]$  is neutral and symmetric for  $T > 2$ .

The physical interpretation of the indefinite metric is borrowed from Dirac's electron theory: The neutral spinor metric  $\beta$  distinguishes input operations from outtake operations (§3.1).

The normed space  $\mathcal{W}[2] = \mathbb{S}[2] \oplus \text{Dual } \mathbb{S}[2]$  of type 2 is a neutral quadratic space of eight dimensions. It completes the first cycle of the Bott periodicity. Its reduced spinor and dual-spinor spaces are also of eight dimensions, and are related to  $\mathcal{W}[2]$  by triality.

One may choose any type  $C$  of  $\mathbb{S}$  to be the  $\psi$  space for an ancestral cell, and analyze any spinor of the event type  $E > C$  as a complex of clones of the cell  $C$ . Then any transformation  $L : \mathbb{S}[C] \rightarrow \mathbb{S}[C]$  induces a transformation  $\sum_C^E L : \mathbb{S}[T] \rightarrow \mathbb{S}[T]$  for all  $T > C$ , the *cumulant* of  $L$ .

If  $X$  is any basic polyadic in  $\mathbb{S}[E]$ , a product of basic monads, then by masking all bars above those of type  $C$  one exhibits a collection of polyads of type  $C$ , whose



transforms under  $L$  are well defined. To transform the complex  $X$ , one transforms each of these type- $C$  polyads in  $X$  in turn and unmask. The sum of all these transforms is  $\sum_C^{C+1} L$ . This is extended to general polyadics by linearity.

In this way any representation of a Lie algebra on the ancestral cell extends to one on the entire complex, its cumulant.  $\mathbb{S}$  is a set theory with null foundation. This construction re-interprets it as a set theory with foundation represented by  $\mathbb{S}[C]$ .

For example, the cell space  $\mathbb{S}[2]$  can be identified with real Dirac spinor space. Then all higher-type spinors of  $\mathbb{S}[T]$ ,  $T > 2$ , represent simplices composed of spins  $1/2$  by iterated Fermi quantification. The Dirac spin vector  $\gamma^m$  for the generic fermion simplex is part of the spin vector  $\gamma^c$  of its ancestral cell. The total spin angular momentum  $J_{c'c}$  is the cumulant over the system complex, of the cell spin angular momenta  $\gamma_{c'c}$  of its constituent cell simplices.

This builds in the spin-statistics correlation as an identity.

The orthogonal group  $\text{SO}(4,4)$  of  $\mathcal{W}[2]$  does not include Yang  $\text{SO}(5,1)$  but it includes the simplicial quantum  $\mathfrak{sl}(4) = \mathfrak{spin}(3,3)$ , which in turn includes Lorentz  $\mathfrak{spin}(3,1)$ .

The two extra timelike dimensions of  $\mathfrak{spin}(3,3)$  are supposed to freeze out in the condensation of  $i$ .

Classical space-time is supposed to emerge as a statistically smoothed approximation to an organized form of such a quantum cellular complex.

Isospin and color, the unitary charges, must commute with spin and orbital angular momentum. They may be represented by operators on the extra 12 spinor dimensions of the ancestral spin cell.

## 2.4 Quantization as quantification

Let the generic one-quantum  $\psi$  be expanded in a monadic basis  $\{1_q\}$  with numerical amplitudes  $\psi^q$ :

$$\psi = \sum_q \psi^q 1_q = \psi^q 1_q. \quad (18)$$

“Second quantization” replaces the numerical coefficients  $\psi^q$  by operators  $\hat{\psi}^q$ . This is often said to misconstrue what is not a quantization at all, since it introduces no quantum constant or homotopy parameter, but a mere quantification, going from one quantum to many, replacing the  $\psi$  vector  $1_q$ , not its numerical coefficient, by an annihilation/creation operator  $\hat{1}_q$ .

But the difference between  $\hat{\psi}^q$  and  $\hat{1}_q$  is a mere duality. This suggests deeper connections between quantization and quantification going both ways:

1. Quantum quantification is also a second quantization in a generalized sense. In a certain singular organized limit, in which the number of quanta  $N \rightarrow \infty$ , a quantum field theory becomes a classical field theory. Were the quantum field theory a true quantization, the return to a classical field theory would be a contraction, a homotopy. In fact the limit  $N \rightarrow \infty$  can be regarded as a generalized contraction, one that proceeds by small discrete steps instead of by a homotopy. In this extended sense, a quantification is also a quantization. The small quantum constant that it introduces is  $1/N$ . In an organized limit  $1/N \rightarrow 0$  the field operators are centralized.

2. Canonical quantization entails a quantification with Bose statistics. To see this, break canonical quantization down into two steps:

1. *Atomize*. Select and decontract a Lie algebra  $a$  of operators, to become the variables of the quantum atomic element.
2. *Quantify*. Form the variables of a quantity of quantum atoms as the polynomials in  $a$  and limits thereof.

The elements of  $\hat{a}$  represent operations that put in and take out quanta of excitation, such as phonons. For the harmonic oscillator, for example,  $a$  is spanned by the elements  $q$  and  $p$ , and is the  $\psi$  space of one phonon. The polynomials in  $p$  and  $q$ , and their limits, correspond to symmetric tensors over  $a$ . The quantized harmonic oscillator is then a bosonic assembly of phonons, a quantified phonon.

The point is that a regular quantum theory requires a quantization based on a regular statistics like the Fermi or Palev statistics. Bose statistics does not work.

The corresponding steps of  $\mathbb{S}$  simplicial quantization are:

1. *Atomize*. Select a graded Lie algebra  $a$  of basic system variables and decontract it into a subalgebra  $\hat{a}$  of  $\mathbb{S}[E] \subset \mathbb{S}$  of simplicial quantum events.
2. *Quantify*. Form the Grassmann algebra in  $\mathbb{S}[E]$  generated by  $\hat{a}$ .

This is a Fermi quantification.

The simplicial quantum theory will use at least six successive quantifications. During the contraction of a simplicial quantum theory to a canonical classical or quantum theory, moreover, the type  $T$  increases beyond any finite bound.

The Grassmann functor  $\bigvee$  quantifies, converting a one-quantum  $\psi$  space to a many-quantum one, and also gauges, cloning one cell Lie algebra into many isomorphs. Any coordinate system  $(x^1, x^2, x^3, x^4)$  orders space-time events into a sequence of sequences of sequences of sequences, implying that events have Maxwell-Boltzmann statistics. No such quanta are found in nature. Fermions and bosons are found, and bosons are singular, so let us assemble space-time as a quantum simplex of simplices of ... of simplices.

Cartan constructed spinors as elements of the exterior algebra over a null “semivector” space. Chevalley noted that Grassmann algebras describe simplicial complexes. Combining their insights, each quantum simplicial cell with  $n$  vertices, is described by a spinor, which we have called a plexor to avoid spacial connotations, and is therefore a spin, or plexon.

The plexor has  $2^n$  components, one for each possible simplicial face. Further structure resides in the deeper levels, covered by one or more bars, successive unitizations. In the singular continuum limit,  $X \rightarrow 0$ ,  $N \rightarrow \infty$ .

In  $\mathbb{S}$ , Grassmann grade counts simplicial quantum events, Quine type counts nested  $\iota$ 's, and the basic type- $E$  dynamical operators  $J^A[E] \in \mathfrak{so}[E]$ , being angular momentum operators of an orthogonal group  $SO[E]$ , count angular momentum in units of the roots of the Lie algebra  $\mathfrak{so}[E]$ .  $\mathfrak{so}[C]$  respects the neutral quadratic form  $\mathfrak{b}[C - 1]$ , which contracts to the Minkowski metrical form.

The action of  $\mathfrak{so}(3,3)$  on type  $C + 1$  is induced by its action on the ancestral cell of type  $C$ . Let us call this familiar summation process and its iterates *cumulation*,

and write it as  $J[C + 1, C] = \Sigma J[C]$ . The action on type  $E$  is then the cumulant  $J[E, C] = \Sigma^{E-C} J[C]$ .

### 3 Simplicial quantum events

At first Einstein described a space-time point or event operationally as a smallest possible occurrence, like the collision of two small hard bodies. It is oddly anachronistic to take collisions of classical macroscopic bodies to be elementary today. Later, events were redefined by their radar coordinates but this concept is still classical, since an electromagnetic wave is actually an unanalyzed photon beam. There seems to be no experimental evidence for an elementary process that defines only a space-time location, as general relativity assumes. In standard theories, nevertheless, fermion events are represented against a classical Minkowski space-time background, by a  $\psi$  vector in an infinite-dimensional complex Hilbert space. Both the Minkowski and Hilbert spaces, having singular groups, must be singular contractions of regular structures closer to experiment. An operational quantum construct of event is still needed.

A single photon is more elementary than a radar signal. The most elementary quantum events to which we have experimental access today are elementary fermion input/outtake operations. Let us take these as the elementary events of the next physics, and regard the various classical event constructs as contracted or truncated descriptions.

These more physical events have coordinates besides position in space-time, forming a non-commutative algebra. According to the Standard Model, fermionic event today carries one hypercharge variable  $y$ , three isospin variables  $\tau^k$ , four space-time position variables  $x^m$ , four momentum-energy variables  $p_m$ , four spin variables  $\gamma^m$ , eight color charges  $\chi^c$ , a ternary generation index  $g$ , and a binary variable distinguishing input from outtake operations. Let us tentatively omit the generation  $g$  for reasons given in §5.

A simplicial quantum theory represents a fermion event by a monadic in a subalgebra  $\mathbb{S}[E] \subset \mathbb{S}$  of some finite type  $E$ ; call this the *event type*. The first-grade event subalgebra  $\mathbb{S}^{(1)}[E]$  replaces Hilbert space as  $\psi$  space to support the algebra  $\text{Alg}[E]$  of single-event operators. Higher grades represent composites of several fermions. The fermion field operators of the Standard Model form a singular Clifford algebra. Those of a simplicial theory are regular Clifford monadics that operate on  $\mathbb{S}[E]$ . They are elementary in that they are not products of non-trivial factors, but they generally have a great many ancestral elements of lower type, since experimental events have high orbital quantum numbers.

The differentiators  $\partial_m$  of canonical quantum theory refer to two levels: An infinite-dimensional orbital level of the variables  $x^m, p_m$  and a four-dimensional tangent-space level with the index  $m$ . The group  $\text{Diff}$  acts on the field and respects the tangent spaces. Both levels are customarily built into the kinematics by restricting the system to differentiable fields and their limits.

In a corresponding class of simplicial quantum theories, monadics  $\hat{1}_c$  of a lower type  $C$  are used as annihilation/creation operators for the foundational elements of which events are composed, and the group of the simplicial quantum theory is required to

respect the partition of the system into cells.

This restriction could be a consequence of the dynamics, not the kinematics. Non-differentiable or cell-breaking transformations may merely require too much energy. The cell level  $C$  might be determined by the action function of the event level, and the kinematics could have the larger group  $\text{spin}(\mathbb{S}[E])$  of the event space. This amounts to assuming that the foundation  $\psi$  space is one-dimensional, and should be kept in mind.

Each  $g$ -adic in  $\mathbb{S}^{(g)}[E]$  is a  $\psi$  for a complex of  $g$  events with Fermi statistics. The singular Lie algebra of Standard Model fermion variables is to be a contraction of a semisimple subalgebra of simplicial quantum event operators in  $\text{Alg}[E]$ .

### 3.1 Bipolar quantum spaces

Dirac compared a negative probability to a bank overdraft [7]. Let us expand this somewhat terse theory of the indefinite metric.

Two frames may share common light cones, and each may draw its future time axis as a vertical arrow, but may still disagree on the distinction between past and future, so that a space-like ray  $r$  that appears to begin at the origin in one frame may appear to end there in another:



If the space-time metric is a dynamical variable, then we cannot restrict our attention to timelike vectors, and an output from the origin can become an input to the origin under this change of frame. Let us take this to indicate that when differential locality breaks down, the distinction between input and output may become relative, like the distinction between past and future.

In general, an operation may introduce an excitation in one circumstance and eliminate one in another. The distinction between input and outtake is then relative to the choice of medium or vacuum.

Let the difference of two normed spaces designate their direct sum provided with the difference norm:

$$\forall Q \in \mathbb{S}, \forall Q' \in \mathbb{S}' : \quad \|Q \ominus Q'\| := \|Q\| - \|Q'\|. \quad (20)$$

Minkoski vectors belong neither to the space  $\mathbb{S}$ , nor to  $\text{Dual } 3Q$ , but to their *bipolar* direct sum

$$\text{Bi } \mathbb{S} := \mathbb{S} \ominus \text{Dual } \mathbb{S}. \quad (21)$$

Each experimental frame now splits the bipolar space  $\mathbb{S}[E]$  into two unipolar subspaces of positive-definite and negative-definite norms respectively. Let us take flux directed from the experimenter into the experiment to be positive. Kets have positive norms and bras have negative norms.

The spinor norm thus gives probability flux rather than net probability. The Lorentz group has regular finite-dimensional isometric representations in the orthogonal group of  $\text{Bi } \mathbb{S}$ .

The ambient spin complex is not random but highly organized locally, into something like a crystal dome whose cell is (say)  $SO(3,3)$  invariant. The dome also supports the particle spectrum, sharp bands of highly coherent transmission.

Each simple Lie algebra of a physical theory has a Killing form with important physical meaning. In the original Yang space, it defines a Minkowski metric on the energy-momentum vector, giving the squared rest-energy. In this way the metrical structure of these quantum space-times derives from their quantum structure.

### 3.2 Simple quantum theories

Nowadays a physical theory involves several Lie or near-Lie algebras, some graded. Let us call a system simple or semisimple if all these algebras are simple or semisimple, respectively, and otherwise *compound*; one bad algebra spoils the barrel. Since the commutative coordinate algebra of a classical system is compound, only a quantum system can be simple. Since neither  $\mathfrak{sl}(\infty)$  nor  $\mathfrak{su}(\infty)$  is a Lie algebra, let alone a simple one, only a quantum system with a finite-dimensional ket space can be simple.

To recover the singular canonical quantum theory (1) from a simple one like (7), one must not only take a singular commutative limit  $\hbar, \mathbf{X} \rightarrow 0$  but must also freeze the degrees of freedom in the quantized imaginary variable  $r = [p, q]$ , so that  $r$  can be treated as a constant  $Ni$ . This freezing can result from a self-organization of many plexons, like that of the spins of a ferromagnet. Let us refer to such a construction briefly as a *singular organized limit*, and write, for example,  $\tilde{q} \succ iq, \tilde{p} \succ ip, \tilde{r}/4N \succ i$ . The symbol  $\hat{x}$  designates a real decontraction of a canonical (classical or quantum) construct  $x \prec \hat{x}$ . The symbol  $\tilde{x}$  indicates a real decontraction of an imaginary canonical construct  $ix \prec \tilde{x}$ .

The lost Hilbert space, canonical commutation relations, and Bose statistics must return in a singular organized limit.

### 3.3 Evolution of simplicial quantum theory

Heisenberg effectively introduced a quantum space for phase space in 1924, and suggested quantizing space-time in 1930 as a way to eliminate the remaining infinities. Julian Wess describes how this idea passed from Heisenberg to Peierls, to Pauli, to Oppenheimer, to Snyder [27], to C. N. Yang [34], who made simplified the Lie algebra of the orbital variables  $x^\mu, p_\mu, L_{\mu'\mu}$  and  $i$  but still represented it in the singular algebra  $\mathfrak{su}(\infty)$  of Hilbert space.

Feynman considered the quantum space (10) in about 1941, before undertaking the Lamb shift.

Independently R. Penrose [20] quantized the Euclidean 2-sphere by representing its points as the directions of the sums of many Pauli spins with Bose-Einstein statistics.

On still another track to quantum space-time, Segal [23] made a suggestion already implicit in Yang's note: that physics evolves toward simple Lie algebras by homotopies that reduce Lie-algebra radicals. Segal's argument was Darwinian: A compound Lie algebra has a singular Killing form, making it labile. A semisimple Lie algebra is stable in this respect. As measurements of the structure constants improve, therefore, a compound Lie algebra has survival probability 0 relative to its semisimple neighbors,

which outnumber it by  $\infty$  to 1. Segal wrote this before the Golden Age of Gauge, but it applies to gauge groups as well. Singular Lie algebras are indeed singular cases; almost all Lie algebras are semisimple.

This Darwinian argument for stabilization must be used with discretion. For example, within the universe of linear spaces with bilinear products the Jacobi identity is as unstable as commutativity, and it is not proposed here to stabilize it. In addition, evolution by small random changes is slow. Simple organizations not only compete but also cooperate, and so form complex organizations faster than random changes can. The fact that relativity and quantum theory are de-radicalizations is sufficient reason to try another.

Gerstenhaber, influenced by Segal, described homologically a rich terrain of Lie algebras connected by homotopies, such as *contractions* [14], that carry groups out of stable valleys of simplicity, to ridges between the valleys, and up to singular peaks [12].

According to the simplicity principle, physics is flowing glacially down the simplicity gradient to a valley in Gerstenhaberland. The Galileo Lie algebra, for example, is on a ridge between the valleys of Lorentz  $\mathfrak{so}(3, 1)$  and the orthogonal group  $\mathfrak{so}(4)$ . Similarly, the Poincaré-canonical Lie algebra  $\mathfrak{alg}(x^m, p_m, L_{m'm}, i)$  is on a ridge between  $\mathfrak{so}(3, 3)$  and  $\mathfrak{so}(5, 1)$ .

Group contraction and deformation quantization were also introduced by others [14, 2].

Vilela Mendez, inspired by Gerstenhaber, rediscovered the Yang  $\mathrm{SO}(5, 1)$  group [31], representing it in Hilbert space, and proposed high-energy physical consequences. The Yang group was rediscovered several times since then.

Galiautdinov [11], Shiri-Garakani [26], and others have studied theories with the Yang  $\mathrm{SO}(5, 1)$  invariance in  $\mathbb{S}$ . Baugh [1] represented the Yang  $\mathfrak{so}(5, 1)$  Lie algebra in  $\mathfrak{sl}(6\mathbb{R})$ .

Now decontraction algorithms are being intensively studied [24].

## 4 Simplicial quantum relativity

Assume that the statistics Lie algebra of simplicial events and the Lie algebra of the  $\psi$  space are simple [19]. Represent  $\mathfrak{spin}(3, 3)$  in  $\mathfrak{spin}(N, N)$ , whose indefinite metric leads to a *bipolar relativity* (§3.1).

Compare the quantized orbital variables of the simple spaces mentioned, with *chPX* units and dimensionless orbital variables.  $\hat{x}$  is a quantized  $x$ ,  $\tilde{x}$  is a quantized  $ix$ ,  $\delta\hat{q}$  is the contribution of one cell to a sum  $\hat{q} = \sum \delta\hat{q}$  over many; and  $1 \leq k \leq 3$ ,  $1 \leq m \leq 4$ :

Feynman	[8]	$\delta\hat{x}^m \sim \gamma^m$	—
Yang	[34]	$\hat{x}^m \sim i(g^{mn}\eta^5\partial_n - \eta^m\partial_5)$	$\hat{p}_m \sim i(\eta_6\partial_m - \eta_m\partial_6)$
Penrose	[20]	$\delta\hat{x}^k \sim \sigma^k$	—
Simplicial	—	$\delta\tilde{x}^m \sim \gamma^{4321}\gamma^m$	$\delta\tilde{p}_m \sim \gamma_m$

(22)

The proposals of Penrose and Feynman maintain an absolute distinction between space-time and energy-momentum. For group simplicity Yang relativized this distinction

within a relativity Lie algebra  $\mathfrak{so}(5, 1)$ , but represented this within the singular Lie algebra  $\mathfrak{su}(\infty)$  of a singular Hilbert space  $L^{(2)}(6\mathbb{R}, \mathbb{C})$ . This respects both reciprocity and parity. For the sake of regularity, the simplicial quantum theory represents both time and energy within a spin representation of Lorentz  $\mathfrak{spin}(3, 1)$  instead. This also introduces fewer variables. Furthermore it violates parity and reciprocity, in better accord with experiment. The factor  $\gamma^{4321}$  has been shifted from  $\delta p$  to  $\delta x$  to make the gaugeon dynamics easier.

Unitary charges of the Standard Model kind are short transverse struts to the dome. The cellular plexons of the ambient dome therefore have a higher dimension than classical space-time, are spins of a higher orthogonal group. Dirac spins are short longitudinal struts. Space-time coordinate lines are longitudinal and long. If they include transverse struts, their unitary charges cancel within a few cells. The quantized imaginary  $\mathring{i}$ —where  $\mathring{\phantom{x}}$  stands for “operator”—is macroscopic and composed of longitudinal struts, and so can be classed as longitudinal, but it is polarized and frozen. There are now three non-zero vacuum expectation values to account for:

- g. Gravity’s  $g_{\mu\nu}$ , which breaks space-time  $\mathfrak{sl}(4\mathbb{R})$ .
- h. Higgs  $h_i$ , which breaks electroweak  $\mathfrak{su}(2)$ .
- i. The  $i$  in the Heisenberg equation and the canonical commutation relations, which breaks complex-plane  $\mathfrak{sl}(2\mathbb{R})$  and time reversal  $\mathbb{Z}_2 \subset \mathrm{SL}(2\mathbb{R})$ .

Let us represent the dome and its excitations within  $\mathbb{S}$  using as few  $\iota$  types as possible.

When  $i$  becomes a variable  $\mathring{i}$  and leaves the center of the algebra, a real quantum theory arises of the Stückelberg kind [29]. This also recalls the Hestenes theory in which  $i = \gamma^{4321}$  [13], and the quaternionic  $i$  that was proposed as a mass-generating Higgs field [30].

The canonical relations work in a segment of the spectrum of  $-\mathring{i}^2$  so near to its maximum value 1 as to be indistinguishable from it in atomic experiments. Yet this narrow band must have a multiplicity that passes today for infinite. For but one example, the band

$$1 - \mathbf{N}^{-1/2} < -\mathring{i}^2 \leq 1 \quad (23)$$

is both narrow and populous enough, with multiplicity  $\mathrm{O}(\sqrt{\mathbf{N}}) \rightarrow \infty$ .

In the singular limit  $\mathring{i} \rightarrow i$ ,  $E \rightarrow 0$ , and (supposedly)  $(\mathring{i})^2 \rightarrow -1$ , the classical space-time and canonical commutation relations are to emerge. We must suppose that

$$\mathbf{XPN} = \hbar, \quad \mathbf{N} \gg 1, \quad h/\mathbf{X} \gg 1 \text{ TeV}, \quad \frac{h}{\mathbf{NX}} \approx 0, \quad (24)$$

in the sense that  $h/\mathbf{NX}$  is presently not resolvable from 0.

To achieve this one may hypothesize an organization akin to polarization that centralizes (“superselects”) the variable  $\mathring{i}$  and contracts it to the imaginary unit of complex quantum theory:  $\mathring{i} \rightarrow i$ . The Heisenberg relation  $[x^m, p_m] = \hbar i$  returns in a singular limit of many cells. This means zero-point estimates based on canonical quantum theory are gross overestimates in the cell domain, including the ones that call for infinite renormalization and energy density.

The coordinates, momenta, and  $\mathring{i}$  are now cumulated spin variables. To make all components of a cumulated spin small is not impossible in the way that making position

and momentum coordinates small is supposed to be. It may merely require a meltdown of  $i$ .

## 4.1 Reciprocity and locality

In canonical classical and quantum theories and all their regularizations considered here, the kinematics obeys reciprocity while experience seems to flout it, especially by its locality. The locality principle permits fields to couple only when they are evaluated at the same position, but different momenta (wave numbers) generally couple.

In the simplicial quantum space (22), reciprocity is the transformation

$$R : \gamma^m \mapsto \gamma^{4321} \gamma^m \quad (25)$$

$R$  is an automorphism of the Dirac Clifford algebra. The simplicial event space has reciprocity. Locality presently breaks it. In the simplicial quantum theory, however, a simplicial “ultra-locality” is tautological: Two simplices  $\gamma^A, \gamma^B$  grade-commute unless they are neighbors, share a vertex  $\gamma^a$ ,  $a \in A, B$ .

## 4.2 Palev statistics

Bose-Einstein statistics is a defining though singular part of canonical quantization. Fermi-Dirac statistics generalizes from Lie algebras to graded Lie algebras, such as Grassmann algebras, in order to deal with variables without classical correspondents, like fermion creators and spins.

A *Palev statistics* (of class  $\mathfrak{a}$ ) is one whose one-quantum annihilation/creation operators generate a semisimple graded Lie algebra  $\mathfrak{a}$ . Fermion pairs are not exactly bosons, only quasi-bosons, but they are exact Palevons. They can simulate bosons as polarized spins can simulate an oscillator by precessing about the polarization axis.

Most attention has been given to the simple ungraded case; but Stoilov and Van der Jeugt find that the Palev statistics of class  $\mathfrak{su}(1|5)$  is especially relevant to the Standard Model [28].

In pure Grassmann simplicial quantum theories, like the present study, one must represent empirical bosons as event dyads. These obey a Palev statistics, with fermionic cores. Presumably this fermion core must show up in high-energy photon-photon collisions. Arguments against composite photons or gravitons are well known, but are based on Bose-Einstein statistics and the canonical commutation relations, and therefore must break down in exactly the domain of interest.

With this generalized concept of statistics, any quantum number can be regarded as a number of quanta. This is familiar for the linear harmonic oscillator, where the quantum is the phonon of excitation.

For instance, the quanta of the dipole rotator in three dimensions, with Lie algebra (7)—call them rotatons, since the term “roton” is preempted—have a statistics that is neither Bose-Einstein nor Fermi-Dirac, but Palev of the  $\mathfrak{spin}(3) = A_1 = B_1$  class. The individual kinds of Palevons correspond to root vectors of the Palevon Lie algebra. The statistics of the Palevon is determined by an irreducible representation of the Palev Lie algebra. Where the canonical commutation relations have just one practical representation, each Palev statistics has infinitely many. The  $\mathfrak{so}(3)$  Palev statistics is



isomorphic to the rotator (7), and has only one kind of quantum and its dual. The Bose-Einstein statistics arises from the representation  $D(N)$  of (7) when  $N$  is allowed to approach infinity with organization that polarizes and centralizes one component, say  $r$ . The corresponding physical theory, however, has a finite  $N$  that must be measured.

The same  $\mathfrak{so}(3)$  class of Palev statistics includes the spinor representation  $D(1/2)$ . Its representation in  $\mathbb{S}$  requires spinors of four real components and is then given by

$$p = \gamma^{32}/2, \quad q = \gamma^{21}/2, \quad r = \gamma^{13}/2. \quad (26)$$

A spin  $1/2$  is thus a composite of up to 2 rotatons with Palev statistics. It is necessary to see how Palev statistics describes a condensation resembling the Bose-Einstein one. This requires many Palevons,  $N \gg 1$ , in at least two classes, one to be filled and the other to be emptied.

A monadic of any type embraces monadics of the type below. As a result, lower-type operators  $L[T]$  induce higher-type ones  $L[T+1] = \Sigma L[T]$ . In particular, quantum spin operators of the ancestral cell induce orbital angular momentum operators of the event. The cumulant  $\Sigma^n L[T]$  represents the operator  $L \in \text{Alg}(\mathbb{S}[T])$  in the algebra  $\text{Alg}(\mathbb{S}[T+n])$  of every higher type  $T+n$ .

Let us apply the cumulation process to the Lie algebra of the simplicial quantum space-time of (22). Let  $\gamma^{\check{b}}$  be four real Dirac spin operators acting on spinors  $1_c$  of a cellular level  $\mathbb{S}[C]$ , with Minkowskian metric form

$$g^{\check{b}'\check{b}} = \frac{1}{2}[\gamma^{\check{b}'}, \gamma^{\check{b}}]. \quad (27)$$

The ancestral group generators  $\delta J_{c'c}[C]$  of the cellular spin are

$$\begin{aligned} \delta \tilde{x}^{\check{b}} &= \mathbf{X} \gamma^{\check{b}}, \quad \check{b} = 1, 2, 3, 4, \\ \delta \tilde{p}_{\check{b}} &= \mathbf{P} \gamma^{4321} \gamma_{\check{b}}, \\ \delta \tilde{i} &= \mathbf{N}^{-1} \gamma^{4321}, \\ \delta \tilde{L}_{\check{b}'\check{b}} &= h \gamma^{4321} \gamma_{\check{b}'\check{b}}, \quad \text{as} \\ \mathfrak{h}(4) &\leftarrow \mathfrak{so}(3, 3) \leftarrow \mathfrak{sl}(6) \end{aligned} \quad (28)$$

The simplicially quantized imaginary  $i$  is normalized to unit magnitude with a factor  $\mathbf{N}^{-1}$ . To form macroscopic monad coordinates, we must cumulate these atomic cell variables at least twice, to reach at least type 6, with dimensionality  $\text{hexp } 6 = 2^{64K}$ , more than ample for a quasi-continuum.

Consider six  $8 \times 8$   $\gamma^{\check{b}}$  of the cellular type  $C$  associated with  $\mathfrak{spin}(3, 3)$ . Choose a frame with metric  $\check{b} = \text{diag}(1, 1, 1, -1, -1, -1)$  and label the  $\gamma$ 's as  $\gamma^1, \dots, \gamma^4, \gamma^X, \gamma^Y$ , with spontaneous polarization of  $\gamma^{YX}$ :  $i = \Sigma^2 \gamma^{YX}[C] \succ i$ . Define the Dirac  $\gamma^m$  as the remaining four first-grade elements of  $\gamma^c[C]$ . They define the Minkowski space-time by cumulation and the singular limit.

### 4.3 The Umklapp problem

When Heisenberg proposed to quantize space-time, Pauli pointed out the Umklapp problem [33]. Feynman raised it for his quantized space-time. Discreteness of the space coordinates with spacing  $\mathbf{X}$ ,

$$x = n\mathbf{X}, \quad (n \in \mathbb{Z}), \quad (29)$$

ordinarily implies periodicity of the momenta with period  $2\pi\mathbf{P} = 2\pi\hbar/\mathbf{X}$ :

$$p \cong p + 2\pi n\mathbf{P}. \quad (30)$$

Then the maximum momentum  $p = \pi n\mathbf{P}$  in one direction is indistinguishable from the maximum momentum  $-p$  in the reverse direction. Phonons in crystals flip their momenta by this process, where the missing momentum is taken up by the crystal; but not photons in the vacuum.

Simplicial quantization creates no Umklapp problem. The spectrum of the momentum  $\mathbf{P}L_{mY}$  is still bounded by  $\pm\mathbf{NP}$ , and is still discrete and uniformly spaced between these bounds, as in a crystal, but it is not uniform in spectral multiplicity, and  $L \doteq \mathbf{N}$  is not the same eigenvector as  $L \doteq -\mathbf{N}$ . None of the variables  $L_{c'c}$  is periodic.

Furthermore, to attain a maximum value for  $L_{mY}$  requires polarizing all  $\mathbf{N}$  spins in the sum along the  $mY$  direction. Since  $L_{mY}$  does not commute with  $L_{YX}$ , this disrupts the dome polarization of the  $L_{XY}$  that produces  $\hat{i}$ , and so must be difficult, perhaps unfeasible.

## 5 Simplicial quantum gauge

Gauge fields are associated with non-commutative fermion momentum variables. A canonical gauge field theory with a semisimple gauge Lie algebra  $\mathfrak{g}$  splits the total momentum-energy  $p_m$  into an invariant kinetic part  $\pi_m$  and a residual potential part  $\Gamma_m$ :

$$p_m = \pi_m + \mathbf{P}\Gamma_m \quad (31)$$

$p_m$  is integrable ( $[p_m, p_{m'}] = 0$ ) but not gauge invariant.  $\pi_m$  is gauge invariant but not integrable.  $\mathbf{P}$  is a constant with dimensions of energy-momentum ( $c = 1$ ).

Simplicial quantization also produces non-commutative momentum components. Since the ancestral atoms of momentum do not commute, neither do their cumulants, the simplicial quantum momenta, the reformations of the singular infinitesimal translations canonical quantum theories. Some of this quantum non-commutativity survives into general relativity as part of the curvature, perhaps including a cosmological constant contribution from the ground complex, a gravitational energy-momentum (“dark energy”). We can be certain that this contribution is finite, and reasonably sure that it is much smaller than the canonical commutation relations permit, since the mean magnitude of the zero-point energy of a simplicial quantum field oscillator is less than its canonical correspondent by a cosmologically large numerical factor.

Therefore it is natural to ask whether gauge fields are simplicial quantum effects expressed in the canonical quantum limit; much as Poisson Brackets are canonical quantum effects expressed in the classical limit. Let us call such a gauge theory simplicial.

In a more canonical theory, the gauge vector fields are “internal” parts of the gravitational field [6]. In a simplicial theory, the gravitational field is an external self-organization of many plexons.

In the simplicial quantum gauge theory, the momentum-energy  $\tilde{p}$ , positional coordinates  $\tilde{x}$ , and quantized imaginary  $\hat{i}$  are generators of  $\mathfrak{sl}(6) \sim \mathfrak{spin}(3, 3)$ , represented as polyadic fermion processes.

Since gaugeons do not obey Fermi statistics they cannot have monadic  $\psi$ 's. The simplest possibility is dyadic, as if they were fermion pairs. Dyadic plexons obey a Palev statistics, and the gauge group is their Palev Lie algebra.

De Broglie suggested that photons are neutrino pairs, a special case. In the canonical quantum theory this seems untenable. According to the Heisenberg uncertainty relations, when two identical fermions are close in their position coordinates they must be far apart in their momenta on the average. The forces binding them would have to be large for high relative momenta. There seem to be no such forces. On the contrary, asymptotic freedom seems to be observed, as well as inferred by Gross and Wilczek from non-abelian quantum gauge theory.

The simplicial quantum gauge theory reopens this classic question by weakening the Heisenberg uncertainty principle enormously, allowing small values of both relative position and relative momentum at once when the organization of  $i$  is highly broken. In the case of a definite gauge metric and an irreducible group, the conservation of the Casimir operator still blocks the possibility that  $p, q, r$  can all be small. But the simplicial quantum metric is indefinite; its Casimir operator can be constant even when its terms decrease.

There are other indications for dyadic simplicial gaugeons:

1 *Spin*. The spins and the statistics add up correctly. This is what motivated De Broglie to propose the fermion-pair theory of the photon and Feynman to consider a fermion-quartet for the graviton. The graviton is the noblest particle, in the way that helium is the noblest gas and the alpha particle a noblest (that is, most magical) nucleus. It would be natural to suppose that, like them, it is composed of four fermions that have saturated each other's valences. Here it is supposed that it and the photon are composed of two fermions that have saturated each other's valences.

The Minkowski index  $m$  of the gaugeon vector  $\Gamma_{m\alpha}(x)$ , where  $\alpha$  indexes a basis for the gauge Lie algebra, contributes  $\hbar$  to the gaugeon spin angular momentum, as usual. The two fermions in the dyad are at slightly different quantum positions and can orbit about each other. The index  $m$  indicates a relative orbital angular momentum of  $1\hbar$ .

The Lie algebra index  $\alpha$  contributes another unit spin for the graviton, where the unitary spins are saturated and  $\alpha$  indexes the gauge Lie algebra  $\mathfrak{spin}(3,1)$ ; and not for the unitary gaugeons, where the fermion spin valences are saturated. Thus all the gaugeons, including the graviton, can be fermion-dual-fermion pairs, plus higher-order terms of more complex structure.

Every fermion has spin  $1/2$  and Fermi-Dirac statistics, and some have color or isospin. It is well-known how to represent fermions in Hilbert space. Then the fermion basis  $\psi$ 's are tensor products of more elementary  $\psi$ 's supporting orbital variables, spin, isospin, color, and generation number. These are its valences. Since they are distinguishable, there is no physical change if Grassmann products are used instead of tensor products. Then the whole construction is readily carried out in  $\mathbb{S}$ ; this adds no insight as yet.

As for the gaugeons: The generic gaugeon has as semisimple gauge group the semi-simple product of all the simple gauge groups of the fermions. If it is a pair composed of just one kind of fermion, the fermion must have everything, so it is a quark. The isospin gaugeons are pairs of quarks in which all the color valences are saturated; the

color gaugeons saturate their isospin valences; gravitons saturate both, leaving only their spins open.

The pair theory of gaugeons has a generational problem, however. There are three generations  $g_1, g_2, g_3$  of fermions, and nine possible pairs  $(g_i g_j)$ , but only one generation of gaugeons is seen.

In one resolution of this conflict, one quark generation  $g_1$  would be more elementary than the other two, and gaugeons are pairs  $g_1 g_1$  of quarks of that generation. The other two generations  $g_2, g_3$  of fermions may then be block from pairing by their structures. In that case, however, they would couple to gaugeons more weakly than the generation  $g_1$ . This would disagree with the Standard Model.

In a resolution that saves the Standard Model, only the symmetric superposition  $g_1 g_1 + g_2 g_2 + g_3 g_3$  is a stable gaugeon. This gaugeon couples to all generations alike.

**2 Connection.** A fermion dyad  $\psi \vee \tilde{\psi}$  (where the tilde indicates the dual particle) indeed transports a third fermion from  $\psi'$  to  $\psi$  as a gauge connection should, according to

$$\psi'' = (\psi \vee \tilde{\psi}') \circ \psi'. \quad (32)$$

**3 Coupling** If the gaugeon is a fermion pair, the Dirac action of the canonical gauge theory must be a contraction of a polyadic simplicial action. Let us de-contract it:

## 5.1 Simplicial gauge dynamics

The Dirac fermion-gaugeon coupling density  $A_D$  is an invariant trilinear form in annihilation/creation operators  $\psi(x)$  for a fermion,  $D_m(x)$  for a gaugeon, and  $\bar{\psi}$  for a dual fermion:

$$A_D(x) = i\bar{\psi}(x)\gamma^m(x)D_m(x)\psi(x) = \succ\!\!\bullet\!\!=, \quad (33)$$

in which “ $\succ$ ” designates the two fermions and “ $=$ ” the gaugeon. All four factors are spin operators with invisible spinor indices, and are multiplied accordingly. Schematically speaking,  $A_D$  couples a vector  $\bar{\psi}\gamma^m\psi$  of the fermion pair to a vector  $D_m$  of the gaugeon at infinitesimally separated space-time points; it is a V-V coupling. Clearly one can form S-S, V-V, T-T, A-A, P-P couplings, and S-P and V-A couplings, as in early theories of beta decay.

The system history  $\psi$  is the time-ordered exponential

$$\Omega = \text{Texp} \int (dx) A_D(x). \quad (34)$$

$A_D$  is required to be skew-hermitian so that  $\Omega$  is unitary.

(27) defines a Clifford algebra only when the gravitational metric  $g^{m'm}(x)$  is treated as a constant quadratic form. When quantum gravity comes into the action, this space-time Clifford algebra breaks down. On the other hand, there is no reason to assume that the Fermi-Dirac Clifford algebra also breaks down, so it is still reasonable to represent quantum gravitational processes with spinors in  $\mathbb{S}$ .

Let us infer from the theories of the other gaugeons that the simplicial gravitational gaugeon too is described by a simplicial correspondent  $\hat{D}$  of the differentiator  $D_m$ , with

curvature field  $[\hat{D}_{m'}, \hat{D}_m] = \hat{R}_{m'm}$ ; not (say) by the simplicial correspondent of the Dirac spin vector  $g^m$ , since (27) breaks down.

Suppose that the gaugeons in  $D_m$  are bound fermionic pairs, associated with two proximate events rather than one, as discussed in §4.2 and §5. Then the Dirac action (33), which couples two fermions and a gaugeon, is actually a surrogate action for a four-fermion (“Fermi”) coupling, effective when two of the fermions are bound; a tetradic

$$A^{(4)} \sim \Gamma^{e'''' e'' e' e} \bar{\psi}_{e'''} \bar{\psi}^{e''} \bar{\psi}_{e'} \bar{\psi}^e \quad (35)$$

in which  $(\Gamma)$  is an invariant coupling tensor with real components.

For a physical theory with this tetradic coupling to be possible, the time development should be an orthogonal operator and its generator  $A^{(4)}$  must be skew-symmetric with respect to the spinor metric form  $\beta$ :

$$(\beta A^{(4)})^\top = -\beta A^{(4)}, \quad (36)$$

where  $(\dots)^\top$  is the transpose of  $(\dots)$ . This holds for four-dimensional spinors of  $\text{spin}(3, 1)$ , whose  $\beta$  is skew-symmetric, if  $(\Gamma)$  has the symmetry property

$$\Gamma^{e'''' e'' e' e} = \Gamma^{e' e'' e'' e}. \quad (37)$$

The simplest-looking tetradic action

$$A^{\text{SS}} \sim (\bar{\psi} \circ \psi)(\bar{\psi} \circ \psi) \quad (38)$$

also has this property; but violates locality grossly, since every event in one pair is coupled to every event in the other. In the Dirac action, only infinitesimally separated events are coupled.

The simplicial action closest to the Dirac action is the V-V coupling

$$A^{\text{VV}} := (\bar{\psi} \gamma^{\check{b}} \psi)(\bar{\psi} \gamma_{\check{b}} \psi) \in \text{Cliff } \mathcal{W}[E] \quad (39)$$

Each factor is a simplicial tensor with invisible indices, whose structure is defined so that it corresponds to that of the Dirac action. Since  $\gamma^{\check{b}}$  puts in or takes out a single vertex, it couples only adjacent plexons, and this may suffice for the experimental locality. A more explicit formulation is reserved for the computational stage.

Turn from fermions to gaugeons. The usual gaugeon action of the Standard Model has the form

$$A_G \sim [D_{m'}, D_m][D^{m'}, D^m]. \quad (40)$$

This becomes an eight-fermion coupling, effective when all eight fermions are bound into four pairs. A corresponding plexon action is octadic:

$$\hat{A}^{(8)} \sim \Gamma^{e^8 e^7 \dots e' e} \bar{\psi}_{e^8} \bar{\psi}^{e^7} \dots \bar{\psi}_{e'} \bar{\psi}^e \quad (41)$$

in which  $(\Gamma)$  is an invariant coupling tensor with real components.

It is impossible to repress the speculation that the octadic gaugeon action  $A^{(8)}$  is an effective action of second order in the tetradic action, operative when eight fermions bind into four pairs. In the continuum-based theory such speculation may be idle, due to infinities, but in the simplicial theory the coefficients, including the fine-structure constant, are mathematically well-defined, and probably computable for sufficiently small-scale experiments.

## 5.2 Space-time curvature as quantum effect

The classical momentum components  $\partial_m$  commute with each other, but the gauge-covariant momentum components  $i\hbar D_m$  do not, nor do the simplicial quantum components  $\hat{p}_m$ . This must contribute to space-time curvature and the gauge tensor fields. It is parsimonious to conjecture that the classical non-commutativity of the gauge differentiator, like that of the Poisson Bracket, is entirely a vestige of the quantum non-commutativity of the pair annihilation/creation operators.

The simplicial quantum curvature is the quantum commutator:

$$\begin{aligned} K_{\tilde{b}'\tilde{b}} &= \mathbf{P}^2[J_{Y\tilde{b}'}, J_{Y\tilde{b}}] \\ &= -\mathbf{P}^2 J_{\tilde{b}'\tilde{b}}, \end{aligned} \tag{42}$$

in a frame where the form  $b$  is in Sylvester normal form with  $b_{YY} = -1$ .  $K$  includes the gravitational curvature  $R$  and the unitary gauge curvatures as non-unified terms in a sum. In the simplicial quantum theory the commutators  $[J_{e'''e''}[E], J_{e'e}[E]]$  are related to other components of the same generating tensor  $J$  by the structure tensor  $c$  of  $\mathbf{spin}[E]$ . Thus the infinity of differential concomitants of the gauge manifold, including the curvature, are replaced in the simplicial quantum theory by the finitely many components of the generating tensor  $J[E]$  of type  $E$ .

## 6 Discussion

Spinors belong to a Grassmann algebra (Cartan); a Grassmann algebra represents a simplicial complex (Chevalley). It has long been suspected [4, 18, 15] that quantum gauge theory might be a singular limit of a simplicial complex theory, and that the gauge charges are vestiges of quantum Burgers(-Volterra) vectors for defects of various kinds. Then the manifold-based gauge theory is not fundamental but a high-level consequence of the underlying cellular dynamics of the complex.

The simplicial quantum theory presented here represents the quantum system as a simplicial quantum complex and its gaugeons, including the graviton, as fermion-dual-fermion pairs. Differential transport is a singular continuum limit of annihilation reactions of the form

$$(\psi \vee \psi'^\beta) \vee \psi'' \rightarrow \psi, \tag{43}$$

where a pair effects a quantum transport by replacing the fermion  $\psi''$  outside the pair by the fermion  $\psi$  inside the pair. The classical principle of gauge invariance is not basic but emerges from the simplicial quantum structure of the deeper levels as a smoothing approximation.

New quantum constants enter: a large integer  $\mathbf{N}$  limiting the number of events, and quantum units  $\mathbf{X}$  and  $\mathbf{E}$  of time and energy. The usual imaginary  $i$  becomes a quantum spin operator, which has to be frozen and centralized by a spontaneous organization in the vacuum. The bras and kets belong to a space  $\mathbb{S}$ , a real Grassmann algebra over itself, not only graded but also typed in the sense of Quine. Its first-grade elements are spinors. Its neutral spinor metric assigns positive norms to kets, negative to bras.

This paper is a palimpsest. Between its lines can dimly be made out remnants of an earlier work called the space-time code [9, 10]. Instead of quantizing the Standard

Model, working from the top down, the space-time code attempted to quantize space-time from the bottom up.

The weakened uncertainty relations for small distances and momenta make it possible that gaugeons, including gravitons, are fermion pairs, as their transformation properties suggest.

A plausible fermion-gaugeon action is then a tetradic (35) in fermion annihilation/creation operators, summed over the cells of the complex. Additional action terms like the Maxwell, Hilbert-Einstein, and Yang-Mills actions for the separate gaugeon fields are not necessary if they are effective actions for organizations derived from (35). Then the gauge coupling constants are order parameters of this organization.

The road from dynamics to particle spectrum and cross-sections is not yet entirely computational. Existing quantum concepts of binding and scattering rest on unstable classical foundations in Minkowski space-time and idealizations like stationary or plane  $\psi$  waves. They too must now be transferred to the more stable quantum foundations of the quantum complex, finite in both the large and the small.

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